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The Additivity of Polygamma Functions

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Abstract. In the paper, the authors prove that the functions $\left|\psi^{(i)}(e^x)\right|$ for $i \in \mathbb{N}$ are subadditive on $(\ln\theta_i, \infty)$ and superadditive on $(-\infty, \ln\theta_i)$, where $\theta_i \in (0,1)$ is the unique root of equation $2\left|\psi^{(i)}(\theta)\right| = \left|\psi^{(i)}(\theta^2)\right|$.

1. Introduction

Recall [5, 7, 9] that a function f is said to be subadditive on I if

$$f(x+y) \le f(x) + f(y)$$

holds for all $x, y \in I$ such that $x + y \in I$. If the above inequality is reversed, then f is called superadditive on the interval I.

The subadditive and superadditive functions play an important role in the theory of differential equations, in the study of semi-groups, in number theory, and also in the theory of convex bodies. A lot of literature for subadditive and superadditive functions can be found in [5, 7, 12, 18] and related references therein.

It is well-known that the classical Euler gamma function $\Gamma(x)$ may be defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \,\mathrm{d}\, t.$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or digamma function, and $\psi^{(k)}(x)$ for $k \in \mathbb{N}$ are called the polygamma functions. It is common knowledge that these functions are fundamental and important and that they have much extensive applications in mathematical sciences.

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In [6], the function $\psi(a+x)$ is proved to be submultiplicative with respect to $x \in [0,\infty)$ if and only if $a \ge a_0$, where a_0 denotes the only positive real number which satisfies $\psi(a_0)=1$.

In [7], the function $[\Gamma(x)]^\alpha$ was proved to be subadditive on $(0,\infty)$ if and only if $\frac{\ln 2}{\ln \Delta} \le \alpha \le 0$, where

$$\Delta = \min_{x\ge 0} \frac{\Gamma(2x)}{\Gamma(x)}.$$

In [3, Lemma 2.4], the function $\psi(e^x)$ was proved to be strictly concave on $\mathbb{R}$.

In [9, Theorem 3.1], the function $\psi(a+e^x)$ is proved to be subadditive on $(-\infty,\infty)$ if and only if $a \ge c_0$, where c_0 is the only positive zero of $\psi(x)$.

In [8, Theorem 1], among other things, it was presented that the function $\psi^{(k)}(e^x)$ for $k \in \mathbb{N}$ is concave (or convex, respectively) on $\mathbb{R}$ if $k=2n-2$ (or $k=2n-1$, respectively) for $n \in \mathbb{N}$.

In [12, 18], some new results on additivity of the remainder of Binet's first formula for the logarithm of the gamma function were established.

For more information on this topic, please refer to [4, 20], especially the monograph [21], and closely related references therein.

In this paper, by employing results in [19], we discuss subadditive and superadditive properties of polygamma functions $\psi^{(i)}(e^x)$ for $i \in \mathbb{N}$.

Our main result may be recited as the following Theorem 1.1.

Theorem 1.1. *The functions* $\left|\psi^{(i)}(e^x)\right|$ *for* $i \in \mathbb{N}$ *are superadditive on* $(-\infty, \ln\theta_i)$ *and subadditive on* $(\ln\theta_i,\infty)$, *where* $\theta_i \in (0,1)$ *is the unique root of equation*

$$2\left|\psi^{(i)}(\theta)\right| = \left|\psi^{(i)}(\theta^2)\right|.$$

2. Proof of Theorem 1.1

Let

$$f_i(x,y) = \left|\psi^{(i)}(x)\right| + \left|\psi^{(i)}(y)\right| - \left|\psi^{(i)}(xy)\right|$$

for $x>0$ and $y>0$, where $i \in \mathbb{N}$. It is clear that $f_i(x,y)=f_i(y,x)$.

In order to show Theorem 1.1, it is sufficient to prove the positivity or negativity of the function $f_i(x,y)$. Direct differentiation yields

$$\frac{\partial f_i(x,y)}{\partial x} = y\left|\psi^{(i+1)}(xy)\right| - \left|\psi^{(i+1)}(x)\right| = \frac{1}{x}\Big[xy\left|\psi^{(i+1)}(xy)\right| - x\left|\psi^{(i+1)}(x)\right|\Big].$$

In [2, Lemma 1] and [10, 19], among other things, the functions $x^\alpha\left|\psi^{(i)}(x)\right|$ are proved to be strictly increasing on $(0,\infty)$ if and only if $\alpha \ge i+1$ and strictly decreasing if and only if $\alpha \le i$. From this monotonicity, it follows easily that

$$\frac{\partial f_i(x,y)}{\partial x} \gtreqless 0$$

if and only if $y \lesseqgtr 1$, which means that the function $f_i(x,y)$ is strictly increasing for $y<1$ and strictly decreasing for $y>1$ in $x \in (0,\infty)$. By the integral representation

$$\psi^{(k)}(x) = (-1)^{k+1}\int_0^\infty \frac{t^k}{1-e^{-t}} e^{-xt} \,\mathrm{d}\, t, \quad x>0, \quad k \in \mathbb{N}$$

in [1, p. 260, 6.4.1], it is easy to see that

$$\lim_{x\to\infty} f_i(x,y) = \left|\psi^{(i)}(y)\right| > 0,$$

then the function $f_i(x,y)$ is positive in $x \in (0,\infty)$ for $y>1$.

For $y<1$, by virtue of the increasing monotonicity of $f_i(x,y)$, it is deduced that

1. if $x > 1$, then
$$f_i(1, y) = \left|\psi^{(i)}(1)\right| < f_i(x, y) < \left|\psi^{(i)}(y)\right|;$$
2. if $x < 1$, then
$$f_i(x, y) < f_i(1, y) = \left|\psi^{(i)}(1)\right|;$$
3. if $y < x < 1$, then
$$f_i(y, y) < f_i(x, y) < f_i(x, x).$$

This implies that
$$f_i(\theta, \theta) = 2\left|\psi^{(i)}(\theta)\right| - \left|\psi^{(i)}(\theta^2)\right| < f_i(x, y)$$
for $y < 1$, where $\theta < 1$ with $\theta < x$ and $\theta < y$.

Using the double inequality
$$\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} < (-1)^{k+1}\psi^{(k)}(x) < \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}}$$
for $x > 0$ and $k \in \mathbb{N}$ in [10, p. 131], [11, Lemma 3], [15], [17, p. 79], and [19, Lemma 3], we obtain
$$f_i(\theta, \theta) < \frac{(i-1)!}{\theta^i}\left[2 + \frac{2i}{\theta} - \frac{1}{\theta^i} - \frac{i}{2\theta^{i+2}}\right] \to -\infty$$
as $\theta \to 0^+$, so
$$\lim_{\theta \to 0^+} f_i(\theta, \theta) = -\infty.$$
Combining this limit with the facts that
$$f_i(1, 1) = \left|\psi^{(i)}(1)\right| > 0$$
and that the function $f_i(\theta, \theta)$ is strictly increasing on $(0, 1)$ yields that the function $f_i(\theta, \theta)$ has a unique zero $\theta_i \in (0, 1)$ such that $f_i(\theta, \theta) > 0$ for $1 > \theta > \theta_i$.

In conclusion, the function $f_i(x, y)$ is positive for $x, y > \theta_i$, and negative for $0 < x, y < \theta_i$. The proof of Theorem 1.1 is complete.

Remark 2.1. Recently, some new properties of the polygamma functions $\psi^{(i)}(x)$ for $i \in \mathbb{N}$ were investigated in the papers [13, 14].

Remark 2.2. This paper is a revised version of the preprint [16].

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